

# Asymptotic zero distribution of complex orthogonal polynomials associated with Gaussian quadrature

A. Deaño<sup>\*</sup>, D. Huybrechs<sup>†</sup>, and A.B.J. Kuijlaars<sup>‡</sup>

## Abstract

In this paper we study the asymptotic behavior of a family of polynomials which are orthogonal with respect to an exponential weight on certain contours of the complex plane. The zeros of these polynomials are the nodes for complex Gaussian quadrature of an oscillatory integral on the real axis with a high order stationary point, and their limit distribution is also analyzed. We show that the zeros accumulate along a contour in the complex plane that has the  $S$ -property in an external field. In addition, the strong asymptotics of the orthogonal polynomials is obtained by applying the nonlinear Deift–Zhou steepest descent method to the corresponding Riemann–Hilbert problem.

## 1 Introduction

### 1.1 Oscillatory integrals

We study the limiting behavior of the zeros of the polynomials that are orthogonal with respect to an oscillatory weight function of exponential type

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<sup>\*</sup>Department of Mathematics, Universidad Carlos III de Madrid, Spain.  
Email: alfredo.deanho@uc3m.es.

<sup>†</sup>Department of Computer Science, Katholieke Universiteit Leuven, Belgium.  
Email: daan.huybrechs@cs.kuleuven.be.

<sup>‡</sup>Department of Mathematics, Katholieke Universiteit Leuven, Belgium.  
Email: arno.kuijlaars@wis.kuleuven.be.

along a path  $\Gamma$  in the complex plane,

$$\int_{\Gamma} \pi_n(z) z^k e^{iz^r} dz = 0, \quad k = 0, 1, \dots, n-1, \quad (1.1)$$

such that the integral is well defined. Parameter  $r \geq 2$  is an integer and we will focus mainly on the case  $r = 3$ .

Our motivation originates in a Fourier-type integral on a finite interval of the real axis of the general form

$$I[f] = \int_a^b f(x) e^{i\omega g(x)} dx, \quad (1.2)$$

where  $\omega > 0$  is a frequency parameter,  $f$  is called the *amplitude* and  $g$  is the *phase* or *oscillator*. Integrals of this type appear in many scientific disciplines involving wave phenomena, such as acoustics, electromagnetics and optics (see for example [14] and references therein). For  $\omega \gg 1$ , integrals of this kind are a recurring topic in asymptotic analysis, and we recall in particular the classical method of steepest descent, which can be applied when  $f$  and  $g$  are analytic in a neighbourhood of  $[a, b]$ , see for instance [27].

We will concentrate on the case where the oscillator  $g$  has a single stationary point  $\xi$  of order  $r-1$  inside the interval  $[a, b]$ , with  $r \geq 2$ , i.e.,  $g^{(j)}(\xi) = 0$ ,  $j = 1, \dots, r-1$  but  $g^{(r)}(\xi) \neq 0$ . Without loss of generality, we take this point  $\xi$  to be the origin and the canonical example is the following:

$$I[f] := \int_a^b f(x) e^{i\omega x^r} dx, \quad (1.3)$$

with  $a < 0$  and  $b > 0$ . Assuming a single stationary point, the general form (1.2) can always be brought into this form by a change of variables.

## 1.2 Numerical evaluation

We assume  $f$  analytic in a complex neighbourhood of the interval  $[a, b]$ . As shown in [15], one possible numerical strategy for the evaluation of (1.3) is to consider paths of steepest descent stemming from the endpoints and from the stationary point. In this way, we can decompose the original integral as follows:

$$\int_a^b f(x) e^{i\omega x} dx = \left( \int_{\Gamma_a} + \int_{\Gamma_0^-} + \int_{\Gamma_0^+} + \int_{\Gamma_b} \right) f(x) e^{i\omega x} dx,$$

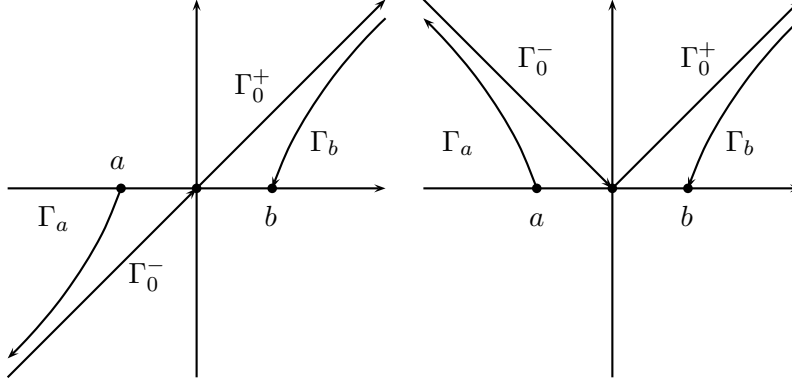


Figure 1: Approximate contours of integration in the complex plane corresponding to even  $r$  (left) and odd  $r$  (right).

where the paths are depicted in Fig. 1. Making an appropriate change of variables, the line integrals along these paths have the form

$$\int_0^\infty u(z) e^{-\omega z^\mu} dz,$$

with  $\mu = 1$  for the endpoints and  $\mu = r$  for a stationary point of order  $r - 1$ . Each of these integrals can be efficiently approximated using Gaussian quadrature, because the optimal polynomial order of Gaussian quadrature translates into optimal asymptotic order in this setting: for  $n$  quadrature points the error behaves like  $\mathcal{O}(\omega^{-\frac{2n+1}{\mu}})$  as  $\omega \rightarrow \infty$ , see [8]. This order is approximately twice that of a classical asymptotic expansion truncated after  $n$  terms.

There are two paths of steepest descent originating from the stationary point, called  $\Gamma_0^-$  and  $\Gamma_0^+$  in Fig. 1. Both paths are straight lines and their structure depends essentially on the parity of  $r$ : for odd  $r$ , these lines form an angle equal to  $\pi - \pi/r$ , whereas for even  $r$  they form one straight line in the complex plane.

In order to keep the total number of function evaluations to a minimum, it is desirable to evaluate both line integrals using only one quadrature rule of Gaussian type [8]. This amounts to constructing a quadrature rule for the functional

$$M[f] = \int_\Gamma f(z) e^{iz^r} dz, \quad (1.4)$$

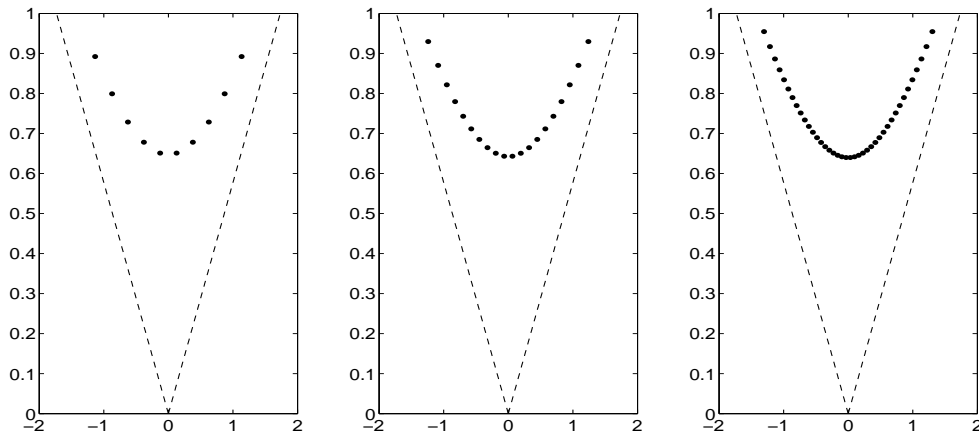


Figure 2: Location of the quadrature nodes for  $r = 3$  on  $[-1, 1]$ , corresponding to  $n = 10$  (left),  $n = 20$  (center) and  $n = 40$  (right). In dashed line, the paths of steepest descent from the origin.

where  $\Gamma = \Gamma_0^- \cup \Gamma_0^+$  is the concatenation of the two steepest descent paths through the origin.

In the case  $r = 2$ , this leads to classical Gauss-Hermite quadrature, which involves the weight function  $e^{-x^2}$  on the real line  $(-\infty, \infty)$ . Higher even values of  $r$  lead to straightforward generalizations, and in all cases the quadrature points lie on the paths of steepest descent.

For odd  $r$ , the functional (1.4) is indefinite and the existence of orthogonal polynomials is not guaranteed a priori. Nevertheless, the orthogonal polynomials and their zeros can be computed numerically. However, one finds that the zeros, which are the complex quadrature points for the integral (3), do not lie on the paths of steepest descent anymore. Instead, they seem to lie on a curve in a sector of the complex plane bounded by the paths of steepest descent. Their location for  $r = 3$  is shown in Fig. 2 for several values of  $n$ . Similar phenomena are observed for larger odd values of  $r$ .

### 1.3 Orthogonality in the complex plane

The problem of Gaussian quadrature leads to the study of polynomials  $\pi_n(z)$  that are orthogonal in the sense of (1.1), where  $r$  is a positive integer ( $r \geq 3$  for a non-classical case) and  $\Gamma$  is the combination of two paths of steepest

descent of the exponential function  $e^{iz^r}$  from the origin, so

$$\arg z = \frac{\pi}{2r} + \frac{2m\pi}{r}, \quad m = 0, 1, \dots, r-1.$$

The straight lines  $\Gamma_0^-$  and  $\Gamma_0^+$  in Fig. 2 correspond respectively to  $m = 0$  and  $m = \lfloor \frac{r}{2} \rfloor$ . In the case  $r = 3$ , these lines form angles of  $\pi/6$  and  $5\pi/6$  with respect to the positive real axis.

Putting  $\lambda_n = (n/r)^{1/r}$  and

$$P_n(z) = \lambda_n^{-n} \pi_n(\lambda_n z) \quad (1.5)$$

we note that (1.1) can be written in the form

$$\int_{\Gamma} P_n(z) z^k e^{-nV(z)} dz = 0, \quad k = 0, \dots, n-1, \quad (1.6)$$

where

$$V(z) = -iz^r/r. \quad (1.7)$$

Note that (1.5) is again a monic polynomial, and the zeros of  $P_n(z)$  and  $\pi_n(z)$  are the same but for rescaling with the parameter  $\lambda_n$ .

The orthogonality (1.6) is an example of non-Hermitian orthogonality with respect to a varying weight on a curve in the complex plane. A basic observation is that the path  $\Gamma$  of the integral in (1.6) can be deformed into any other curve that is homotopic to it in the finite plane, and that connects the same two sectors at infinity. For any such deformed  $\Gamma$  we still have the orthogonality condition (1.6).

In order to find where the zeros of  $P_n(z)$  lie for large  $n$ , we have to select the ‘right’ contour. Stahl [25] and Gonchar–Rakhmanov [13, Sec. 3] studied and solved this problem, and from their works it is known that the appropriate contour should have a symmetry property (the so-called *S*-property) in the sense of logarithmic potential theory with external fields. We recall this concept in the next subsection.

In the case (1.7) with  $r = 3$ , we can identify the curve with the *S*-property explicitly as a critical trajectory of a quadratic differential. Other cases where the potential problem is explicitly solved include [13], [21] and [2] in connection with best rational approximation of  $e^{-x}$  on  $[0, \infty)$ , [4] in connection with a last passage percolation problem, and [18], [19], [17], [22], [23] in connection with classical orthogonal polynomials (Laguerre and Jacobi) with non-standard parameters.

Varying orthogonality on complex curves is treated in detail in the more recent accounts [3] and [7], which contain a Riemann-Hilbert steepest descent analysis in a fairly general setting, assuming the knowledge of the curve with the  $S$ -property. See also [6] for an approach based on algebraic geometry and Boutroux curves and [5] for extensions to  $L_p$  optimal polynomials.

## 1.4 The $S$ property

Let  $V$  be a polynomial. We consider a smooth curve  $\Gamma \subset \mathbb{C}$ , such that the integral in (1.6) is well-defined, and we want to minimize the weighted energy:

$$I_V(\nu) = \iint \log \frac{1}{|z-s|} d\nu(z) d\nu(s) + \operatorname{Re} \int V(s) d\nu(s), \quad (1.8)$$

among all Borel probability measures  $\nu$  supported on  $\Gamma$ . Following the general theory of logarithmic potential theory with external fields, see [24], this problem has a unique solution, which is called the equilibrium measure on  $\Gamma$  in the presence of the external field  $\operatorname{Re} V$ . We denote this equilibrium measure by  $\mu$ .

Let

$$U^\mu(z) = \int \log \frac{1}{|z-s|} d\mu(s) \quad (1.9)$$

be the logarithmic potential of  $\mu$ . It satisfies

$$\begin{aligned} 2U^\mu(z) + \operatorname{Re} V(z) &= \ell, & z \in \operatorname{supp} \mu, \\ 2U^\mu(z) + \operatorname{Re} V(z) &\geq \ell, & z \in \Gamma \setminus \operatorname{supp} \mu, \end{aligned} \quad (1.10)$$

for some constant  $\ell$ , see [24]. If  $\Gamma$  is an analytic contour, then  $\operatorname{supp} \mu$  will consist of a finite union of analytic arcs. Now we can define the  $S$ -property.

**Definition 1.1.** *The analytic contour  $\Gamma$  has the  $S$ -property in the external field  $\operatorname{Re} V$  if for every  $z$  in the interior of the analytic arcs that constitute  $\operatorname{supp} \mu$ , we have*

$$\frac{\partial}{\partial n_+} [2U^\mu(z) + \operatorname{Re} V(z)] = \frac{\partial}{\partial n_-} [2U^\mu(z) + \operatorname{Re} V(z)]. \quad (1.11)$$

Here  $\frac{\partial}{\partial n_\pm}$  denote the two normal derivatives taken on either side of  $\Gamma$ .

The result of Gonchar-Rakhmanov then reads (for the special case of polynomial  $V$ ):

**Theorem 1.2. *Gonchar-Rakhmanov [13, Sec. 3]*** *If  $\Gamma$  is a contour with the  $S$ -property (1.11) in the external field  $\operatorname{Re} V$ , then the equilibrium measure  $\mu$  on  $\Gamma$  in the external field  $\operatorname{Re} V$  is the weak limit of the normalized zero counting measures of the polynomials  $P_n$  defined by the orthogonality (1.6).*

## 1.5 Outline of the paper

In the next section we present the main results of this paper, corresponding to (1.7) with  $r = 3$ . These can be summarized in the following points:

- We present a finite curve  $\gamma \subset \mathbb{C}$ , which is a critical trajectory of a certain quadratic differential  $Q(z)dz^2$ , see Theorem 2.1.
- We prove that this curve  $\gamma$  can be prolonged to  $\infty$  in a suitable way, thus obtaining a curve  $\Gamma$  with the  $S$ -property in the presence of the external field  $\operatorname{Re} V$ , see Theorem 2.2.
- As a consequence of Gonchar-Rakhmanov theorem, it is possible to obtain the weak limit distribution of the zeros of  $P_n(z)$  as  $n \rightarrow \infty$ , see Theorem 2.3.
- Additionally, a full Riemann–Hilbert analysis of this problem is feasible and yields both existence of the sequence of orthogonal polynomials  $P_n(z)$  for large enough  $n$  and the asymptotic behavior of  $P_n(z)$  in various regions of the complex plane as  $n \rightarrow \infty$ , see Theorem 2.4.

## 2 Statement of results

### 2.1 Definition of the curve $\gamma$

In the case (1.7) with  $r = 3$ , the curve with the  $S$ -property is given in terms of the critical trajectory of the quadratic differential  $Q(z)dz^2$ , where

$$Q(z) = -\frac{1}{4}(z+i)^2(z^2-2iz-3). \quad (2.1)$$

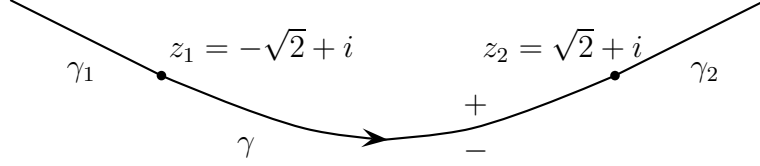


Figure 3: The contour  $\Gamma$  consists of the critical trajectory  $\gamma$  and its analytic continuations  $\gamma_1$  and  $\gamma_2$ .

The polynomial (2.1) has a double root at  $z = -i$  and two simple roots at  $z_1 = -\sqrt{2} + i$  and  $z_2 = \sqrt{2} + i$ .

The critical trajectory  $\gamma$  is an analytic arc from  $z_1$  to  $z_2$  so that

$$\frac{1}{\pi i} \int_{z_1}^z Q^{1/2}(s) ds \quad (2.2)$$

is real for  $z \in \gamma$ , see [26]. We first show that this curve indeed exists.

**Theorem 2.1.** *There exists a critical trajectory  $\gamma$  of the quadratic differential  $Q(z)dz^2$ , where  $Q(z)$  is given in (2.1), that connects the two zeros  $z_1 = -\sqrt{2} + i$  and  $z_2 = \sqrt{2} + i$  of  $Q$ .*

The proof of the theorem is contained in Section 3.

## 2.2 Contour with $S$ -property

In what follows we use the analytic arc  $\gamma$ , whose existence is guaranteed by Theorem 2.1, with an orientation so that  $z_1$  is the starting point of  $\gamma$  and  $z_2$  is the ending point. The  $+$  side ( $-$  side) of  $\gamma$  is on the left (right) as we traverse  $\gamma$  according to its orientation, as shown in Figure 3.

From now on the square root  $Q^{1/2}(z)$  is defined with a branch cut along  $\gamma$  and so that

$$Q^{1/2}(z) = -\frac{1}{2}iz^2 - \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \quad (2.3)$$

as  $z \rightarrow \infty$ . This branch is then used for example in (2.2). We use  $Q_+^{1/2}(s)$ , when  $s \in \gamma$ , to denote the limiting value of  $Q^{1/2}(z)$  as  $z$  approaches  $s \in \gamma$  from the  $+$  side.



The curve  $\gamma$  has an analytic extension to an unbounded oriented contour

$$\Gamma = \gamma_1 \cup \gamma \cup \gamma_2 \quad (2.4)$$

that we use for the orthogonality (1.6). The parts  $\gamma_1$  and  $\gamma_2$  are such that

$$\phi_1(z) = \int_{z_1}^z Q^{1/2}(s)ds \quad \text{is real and positive for } z \in \gamma_1, \quad (2.5)$$

and

$$\phi_2(z) = \int_{z_2}^z Q^{1/2}(s)ds \quad \text{is real and positive for } z \in \gamma_2. \quad (2.6)$$

The main result of the paper is then the following.

**Theorem 2.2.** *The contour  $\Gamma$  is a curve with the  $S$ -property in the external field  $\operatorname{Re} V$ . In addition, we have that the equilibrium measure on  $\Gamma$  in the external field  $\operatorname{Re} V$  is given by the probability measure*

$$d\mu(s) = \frac{1}{\pi i} Q_+^{1/2}(s)ds. \quad (2.7)$$

The proof of this theorem is presented in Section 4. The general result of Gonchar-Rakhmanov, see Theorem 1.2, then implies:

**Theorem 2.3.** *Assume  $r = 3$ . For large enough  $n$  the monic polynomial  $P_n(z)$  of degree  $n$  satisfying the orthogonality relation (1.6) with  $V(z)$  given by (1.7) exists uniquely. Furthermore, denoting by*

$$z_{1,n}, \dots, z_{n,n}$$

*its  $n$  zeros in the complex plane, we have*

- (a) *as  $n \rightarrow \infty$  the zeros accumulate on  $\gamma$ ;*
- (b) *the normalized zero counting measures have a weak limit*

$$\frac{1}{n} \sum_{k=1}^n \delta_{z_{k,n}} \xrightarrow{*} d\mu,$$

*where  $\mu$  is given by (2.7).*

## 2.3 Riemann-Hilbert analysis

Theorem 2.3 also follows from a steepest descent analysis for the Riemann-Hilbert problem that characterizes the polynomials  $P_n$ . We include an exposition of this method in Section 5 for two main reasons:

- The Riemann-Hilbert analysis not only provides the limit behavior of the distribution of zeros of  $P_n(z)$ , but also strong asymptotics of the orthogonal polynomials in the complex plane.
- The present case can be viewed as a simple model problem for Riemann-Hilbert analysis in the complex plane. We hope that the present analysis can be useful as an introduction to this powerful method.

The Riemann-Hilbert problem for orthogonal polynomials was found by Fokas, Its and Kitaev [12], and the steepest descent analysis of Riemann-Hilbert problems is due to Deift and Zhou [11]. The steepest descent analysis for orthogonal polynomials with varying weights on the real line is due to Deift et al., see [10] and [9].

The extension of this method to orthogonal polynomials on curves in the complex plane is not new. It has already been presented in various papers, see for example [2], [3], [7], [10] and [18]. However, an attractive feature of the example treated here is that all quantities in the analysis can be computed explicitly. In that respect it is similar to [18].

In order to formulate the additional asymptotic results that follow from the steepest descent analysis, we introduce some more notation. We use the function  $\phi_2(z)$  defined in (2.6) and the related function (the  $g$ -function)

$$g(z) = \frac{1}{2}V(z) - \phi_2(z) - l, \quad l = \frac{1}{3} + \frac{1}{2}\log 2, \quad (2.8)$$

which has an alternative expression

$$g(z) = \int \log(z - s)d\mu(s)$$

in terms of the equilibrium measure  $\mu$  on  $\gamma$ . In the case  $r = 3$  there is an explicit expression for  $\phi_2(z)$ :

$$\begin{aligned} \phi_2(z) = & -\frac{i}{6}z(z+i)\sqrt{z^2-2iz-3} \\ & - \log(z-i+\sqrt{z^2-2iz-3}) + \frac{1}{2}\log 2. \end{aligned} \quad (2.9)$$

The so-called global parametrix  $N(z)$  is defined in terms of the function

$$\beta(z) = \left( \frac{z - z_2}{z - z_1} \right)^{1/4}, \quad z \in \mathbb{C} \setminus \gamma, \quad (2.10)$$

with the branch cut taken along  $\gamma$ .  $N(z)$  is a  $2 \times 2$  matrix valued function with entries

$$N_{11}(z) = N_{22}(z) = \frac{\beta(z) + \beta(z)^{-1}}{2}, \quad N_{12}(z) = -N_{21}(z) = \frac{\beta(z) - \beta(z)^{-1}}{2i},$$

that also appear in the asymptotic formulas in Theorem 2.4.

Finally, near the endpoint  $z_2$  we require a conformal map

$$f(z) = \left[ \frac{3}{2} \phi_2(z) \right]^{2/3}, \quad (2.11)$$

which maps  $\gamma$  and  $\gamma_2$  near  $z_2$  into the real line. A local Riemann-Hilbert problem is solvable explicitly in terms of the usual Airy function  $\text{Ai}(z)$  and its derivative  $\text{Ai}'(z)$ . These functions appear in the asymptotic formula in part (c) of Theorem 2.4 that is valid in a neighborhood of  $z_2$ .

The steepest descent analysis of the Riemann-Hilbert problem then leads to the following result:

**Theorem 2.4.** *Assume  $r = 3$ . Let  $U_\delta(z_1)$  and  $U_\delta(z_2)$  be small neighbourhoods of the points  $z_1$  and  $z_2$  given before. As  $n \rightarrow \infty$  the polynomial  $P_n(z)$  has the following asymptotic behavior:*

(a) *Uniformly for  $z$  in compact subsets of  $\overline{\mathbb{C}} \setminus \gamma$ , we have*

$$P_n(z) = N_{11}(z) e^{ng(z)} (1 + O(1/n)) \quad (2.12)$$

*as  $n \rightarrow \infty$ ;*

(b) *There is a neighbourhood  $U$  of  $\gamma$  in the complex plane, so that, uniformly for  $z \in U \setminus (U_\delta(z_1) \cup U_\delta(z_2))$ :*

$$P_n(z) = e^{n[\frac{V(z)}{2} - l]} (e^{-n\phi_2(z)} N_{11}(z) \pm i e^{n\phi_2(z)} N_{12}(z) + O(1/n)), \quad (2.13)$$

*where the  $+$  ( $-$ ) sign in (2.13) is valid for  $z$  in the part of  $U$  that lies above (below) the curve  $\gamma$ ;*

(c) *Uniformly for  $z \in U_\delta(z_2)$ , we have as  $n \rightarrow \infty$ ,*

$$P_n(z) = \sqrt{\pi} e^{n[\frac{V(z)}{2} - l]} \left( n^{1/6} f^{1/4}(z) \beta^{-1}(z) \operatorname{Ai}(n^{2/3} f(z)) (1 + \mathcal{O}(1/n)) \right. \\ \left. - n^{-1/6} f^{-1/4}(z) \beta(z) \operatorname{Ai}'(n^{2/3} f(z)) (1 + \mathcal{O}(1/n)) \right),$$

*with the same constant  $l$  as in (2.8).*

### 3 Proof of Theorem 2.1

It follows from the general theory, see [26], that three trajectories of the quadratic differential  $Q(z)dz^2$  emanate from each simple zero of  $Q$ . The three trajectories through  $z_1$  emanate from  $z_1$  at angles  $\theta$  that satisfy

$$3\theta = \pi - Q'(z_1) \pmod{2\pi}.$$

From the explicit formula of  $Q$  and  $z_1$  we find  $Q'(z_1) = -\sqrt{2} - 4i$  and the three angles at  $z_1$  are

$$\theta = -\frac{1}{3} \arctan(2\sqrt{2}) + \frac{2k\pi}{3}, \quad k = 0, 1, 2.$$

We let  $\gamma$  be the trajectory that emanates from  $z_1$  at angle

$$\theta_0 = -\frac{1}{3} \arctan(2\sqrt{2}) = -0.4103 \dots$$

Let

$$D(z) = \frac{1}{\pi i} \phi_1(z) = \frac{1}{\pi i} \int_{z_1}^z Q^{1/2}(s) ds, \quad z \in \mathbb{C} \setminus \gamma.$$

Figure 4 shows the level curves  $\operatorname{Re} D(z) = 0$ ,  $\operatorname{Re} D(z) = 1$ , and  $\operatorname{Im} D(z) = 0$ . In order to prove that  $z_1$  and  $z_2$  are indeed connected by  $\gamma$  (as suggested by the figure), we use arclength parametrization of  $\gamma$

$$\gamma : \quad z = z(t), \quad z(0) = z_1.$$

Then

$$\int_{z(0)}^{z(t)} Q^{1/2}(s) ds = \pi i f(t),$$

where  $f(t)$  is real. Differentiating and squaring, we obtain

$$Q(z(t))[z'(t)]^2 = -\pi^2 [f'(t)]^2,$$

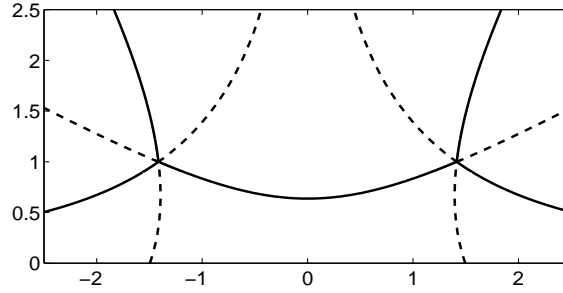


Figure 4: Contour lines of  $D(z)$  in the cubic case. In solid line  $\text{Im } D(z) = 0$ , and in dashed line,  $\text{Re } D(z) = 0$  and  $\text{Re } D(z) = 0$  (left and right of the figure respectively).

which implies that

$$\arg Q(z(t)) + 2 \arg z'(t) = \pi \pmod{2\pi}.$$

The level lines  $\text{Re } Q(z) = 0$  and  $\text{Im } Q(z) = 0$  are shown in Fig. 5. The two level lines intersect of course at the zeros  $z_1$ ,  $z_2$  and  $-i$  of  $Q$ . The critical trajectory  $\gamma$  starts at  $z_1$  at an angle  $\theta_0 = -\frac{1}{3} \arctan(2\sqrt{2})$ , and therefore  $\gamma$  enters the shaded region of Fig. 5, which is the region where  $\text{Re } Q(z) < 0$  and  $\text{Im } Q(z) < 0$ , hence  $-\pi < \arg Q(z) < -\pi/2$ . As a consequence we have that

$$-\frac{\pi}{4} < \arg z'(t) < 0$$

as long as  $z(t)$  is in the shaded region. This implies that the real part of  $z(t)$  increases faster than the imaginary part decreases. It follows that the part of  $\gamma$  that is in the shaded region is contained in the triangle with vertices  $z_1 = -\sqrt{2} + i$ ,  $i$  and  $(1 - \sqrt{2})i$ . Hence  $\gamma$  leaves the shaded region at a point on the imaginary axis above the other critical point  $z_0 = -i$ . Then by the symmetry with respect to the imaginary axis we conclude that  $\gamma$  indeed connects  $z_1$  and  $z_2$ .

This completes the proof of Theorem 2.1.

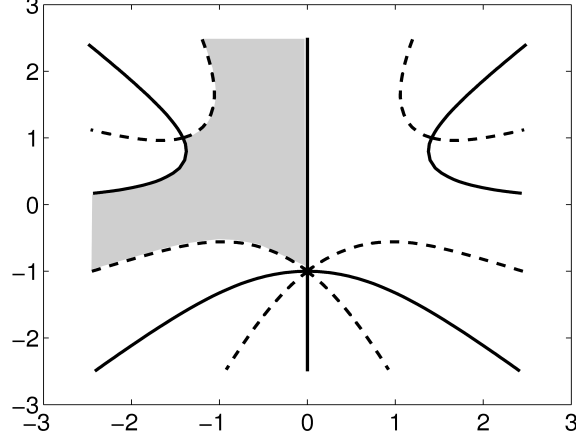


Figure 5: Contour lines of  $Q(z)$  in the cubic case. In solid line  $\operatorname{Re} Q(z) = 0$ , and in dashed line  $\operatorname{Im} Q(z) = 0$ .

## 4 Proof of Theorem 2.2

We start by presenting a characterization of the curve  $\Gamma$  that is equivalent to the  $S$ -property.

For complex  $z$ , we define the  $g$ -function

$$g(z) = \int_{\Gamma} \log(z - s) d\mu(s), \quad (4.1)$$

which is analytic when  $z \in \mathbb{C} \setminus \Gamma$ . We observe that  $\operatorname{Re} g(z) = -U^{\mu}(z)$ , where  $U^{\mu}$  is the logarithmic potential as defined in (1.9). The equilibrium properties (1.10) then translate into

$$\begin{aligned} \operatorname{Re}(-g_{+}(z) - g_{-}(z) + V(z)) &= \ell, & z \in \operatorname{supp} \mu, \\ \operatorname{Re}(-g_{+}(z) - g_{-}(z) + V(z)) &\geq \ell, & z \in \Gamma \setminus \operatorname{supp} \mu. \end{aligned} \quad (4.2)$$

Let us write  $\gamma = \operatorname{supp} \mu$ , then from the Cauchy-Riemann equations it follows that the  $S$ -property (1.11) is equivalent to the property that the imaginary part of  $-g_{+} - g_{-} + V$  is locally constant on  $\gamma$ , that is

$$\operatorname{Im}(-g_{+}(z) - g_{-}(z) + V(z)) = \tilde{\ell}, \quad z \in \gamma \quad (4.3)$$

with a possibly different constant  $\tilde{\ell}$  on the different components of  $\gamma$ . Then as a consequence of (4.2) and (4.3) we have that

$$-g_+(z) - g_-(z) + V(z) = \ell + i\tilde{\ell} \quad (4.4)$$

is constant on each connected component of  $\gamma$ . Differentiating (4.4) we obtain

$$-g'_+(z) - g'_-(z) + V'(z) = 0, \quad z \in \gamma. \quad (4.5)$$

Next we observe that the function  $\frac{1}{2}V'(z) - g'(z)$  is analytic for  $z \in \mathbb{C} \setminus \gamma$ , and furthermore using (4.5):

$$\left(\frac{1}{2}V'(z) - g'(z)\right)_+ = -\frac{1}{2}V'(z) + g'(z)_- = -\left(\frac{1}{2}V'(z) - g'(z)\right)_-$$

for  $z \in \gamma$ . Hence  $\frac{1}{2}V'(z) - g'(z)$  has a multiplicative jump of  $-1$  on  $\gamma$ , and therefore

$$Q(z) := \left(\frac{1}{2}V'(z) - g'(z)\right)^2 \quad (4.6)$$

is analytic in the whole complex plane. The asymptotic behavior of  $Q(z)$  for  $z \rightarrow \infty$  follows from the fact that  $V(z)$  is a polynomial and

$$g'(z) = \int \frac{1}{z-s} d\mu(s) = \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \quad z \rightarrow \infty, \quad (4.7)$$

since  $\mu$  is a probability measure on  $\gamma$ . The general case of Liouville's theorem implies that  $Q$  is a polynomial of degree  $2r - 2$  if  $\deg V = r$ .

In general this is not enough to determine the curve  $\gamma$ , and we need more information on the roots of  $Q(z)$  (or extra assumptions, such as that we are in the one-cut case). Since  $Q^{1/2}(z)$  is analytic in  $\mathbb{C} \setminus \gamma$  we can deduce that any zero of odd multiplicity of  $Q$  is in  $\gamma$ . Zeros of even multiplicity can be anywhere and are typically not in  $\gamma$ .

Let  $z_1$  be a zero of  $Q$  of odd multiplicity. From (4.7) we see that for  $z \in \gamma$ , in the same connected component as  $z_1$ , we have

$$\int_{z_1}^z Q_+^{1/2}(s) ds \in i\mathbb{R}.$$

This is the condition that characterizes a trajectory of the quadratic differential  $Q(z)dz^2$ , emanating from a zero  $z_1$  of  $Q$ .

In the case  $V(z) = -iz^3/3$ , it follows from (4.6) and (4.7) that  $Q$  should be taken as a polynomial of degree 4 so that

$$Q(z) = \left( -\frac{iz^2}{2} - \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \right)^2 = -\frac{z^4}{4} + iz + C, \quad (4.8)$$

where  $C$  needs to be determined. In order to do this, we make the assumption (to be justified later) that we are in the one-cut case, that is we assume that  $\gamma$  is a single curve. The endpoints of the curve are then simple zeros of  $Q$ . Since  $Q$  has degree four there are two more zeros, which in the one-cut case, should combine into a double zero.

In our case, there is a symmetry about the imaginary axis, and therefore the double root should be on the imaginary axis, say  $z = z_0$ , and two simple roots are symmetric with respect to the imaginary axis, say  $z_1$  and  $z_2 = -\bar{z}_1$ . This leads to

$$Q(z) = -\frac{1}{4}(z - z_0)^2(z - z_1)(z + \bar{z}_1),$$

which combined with (4.8) yields

$$z_0 = -i, \quad \text{and} \quad z_1 = -\sqrt{2} + i, \quad z_2 = \sqrt{2} + i.$$

The free constant is  $C = -3/4$ . Therefore

$$Q(z) = -\frac{1}{4}(z + i)^2(z^2 - 2iz - 3), \quad (4.9)$$

and we recover (2.1).

Once we have  $Q(z)$ , we may obtain  $\mu$  in the following way. From (4.6) it follows that there is an analytic branch of  $Q^{1/2}(z)$  for  $z \in \mathbb{C} \setminus \gamma$  which behaves as  $\frac{1}{2}V'(z)$  for large  $z$ . Choose an orientation on  $\gamma$ . The orientation induces a  $+$ -side and a  $-$ -side on  $\gamma$ , where the  $+$ -side ( $-$ -side) is on the left (right) as one traverses the contour according to its orientation.

**Lemma 4.1.** *Given the critical trajectory  $\gamma$  and the polynomial  $Q(z)$ , then*

$$\frac{1}{\pi i} Q_+^{1/2}(s) ds = d\mu \quad (4.10)$$

*is a probability measure on  $\gamma$ .*



*Proof.* The measure  $\mu$  is a priori complex, however, by the construction of  $\gamma$  we have that

$$\int_{z_1}^z d\mu(s) = \frac{1}{\pi i} \int_{z_1}^z Q_+^{1/2}(s) ds \in \mathbb{R}$$

for every  $z \in \gamma$ , so that  $\mu$  is a real measure.

Taking  $z = z_2$  we can compute

$$\int_{z_1}^{z_2} d\mu(s) = \frac{1}{\pi i} \int_{z_1}^{z_2} Q_+^{1/2}(s) ds$$

by contour integration. Indeed, we have

$$\frac{1}{\pi i} \int_{z_1}^{z_2} Q_+^{1/2}(s) ds = \frac{1}{2\pi i} \int_C Q^{1/2}(s) ds$$

where  $C$  is a closed contour in  $\mathbb{C} \setminus \gamma$  that encircles  $\gamma$  once in the clockwise direction. Moving the contour to infinity, and using the behavior of  $Q^{1/2}$  at infinity, see (2.3), we find

$$\mu(\gamma) = \int_{z_1}^{z_2} d\mu(s) = 1. \quad (4.11)$$

Then if  $t \in [0, 1] \mapsto z = z(t)$  is a smooth parametrization of  $\gamma$  with  $z(0) = z_1$  and  $z(1) = z_2$  we have that

$$t \in [0, 1] \mapsto \frac{1}{\pi i} \int_{z_1}^{z(t)} Q_+^{1/2}(s) ds$$

is real valued, with values 0 for  $t = 0$  and 1 for  $t = 1$ . The derivative  $z'(t)Q^{1/2}(z(t))$  is non-zero for  $0 < t < 1$ . Therefore

$$\frac{1}{\pi i} \int_{z_1}^{z(t)} Q_+^{1/2}(s) ds$$

is strictly increasing, and it follows that  $\mu$  is a probability measure.  $\square$

**Lemma 4.2.** *Let  $\Gamma = \gamma_1 \cup \gamma \cup \gamma_2$  be defined as in (2.4), (2.5) and (2.6), then the measure  $\mu$  defined by (4.10) is the equilibrium measure on  $\Gamma$  in the external field  $\operatorname{Re} V$ .*

*Proof.* From another residue calculation, similar to the one leading to (4.11) and based on (2.3), it follows that

$$\int_{\gamma} \frac{1}{z-s} d\mu(s) = \frac{1}{2} V'(z) - Q^{1/2}(z), \quad z \in \mathbb{C} \setminus \gamma. \quad (4.12)$$

Then we have

$$g(z) = \int \log(z-s) d\mu(s)$$

is such that (4.5) holds, which after integration leads to (4.3) and to the first line of (4.2).

We extend  $\gamma$  to an unbounded contour  $\Gamma = \gamma \cup \gamma_1 \cup \gamma_2$  as in section 2.2. The unbounded pieces  $\gamma_1$  and  $\gamma_2$  are such that (2.5) and (2.6) hold. This leads to the second line of (4.2). For example, if  $z \in \gamma_2$ , then by (2.6) and (4.12)

$$\begin{aligned} 0 < 2 \int_{z_2}^z Q^{1/2}(s) ds &= \int_{z_2}^z (V'(s) - 2g'(s)) ds \\ &= (V(z) - 2g(z)) - (V(z_2) - 2g(z_2)) \\ &= V(z) - 2g(z) - (\ell + i\tilde{\ell}), \end{aligned}$$

which by taking the real part indeed leads to the inequality in (4.2).

Because of (4.2) we have that  $\mu$  is the equilibrium measure on  $\Gamma$  in the external field  $\operatorname{Re} V$ .  $\square$

Finally, because of (4.3) we conclude that the contour  $\Gamma$  has the  $S$ -property, and this completes the proof of Theorem 2.2.

## 5 Proof of Theorem 2.4

### 5.1 Riemann–Hilbert problem

The orthogonal polynomial  $P_n(z)$  characterized by (1.6) appears as the  $(1, 1)$  entry of the solution  $Y(z)$  of a  $2 \times 2$  matrix-valued Riemann–Hilbert problem, see [12].

From this Riemann–Hilbert problem, the Deift–Zhou steepest descent method performs several explicit and invertible transformations that allow us to obtain asymptotic results for the entries of the matrix  $Y$ , and in particular for  $P_n(z)$ , as  $n \rightarrow \infty$  uniformly in different regions of  $\mathbb{C}$ , see [10]. In

the present case the analysis is quite standard, except for the fact that we are working on a complex curve  $\Gamma$  instead of on a part of the real line. For this reason, we give a brief sketch of the method and refer the reader to [10], [9] and [18] for the general theory involving orthogonality with respect to exponential weights and also for more details on a similar problem.

We are interested in a matrix-valued function  $Y : \mathbb{C} \setminus \Gamma \rightarrow \mathbb{C}^{2 \times 2}$  such that

- $Y(z)$  is analytic for  $z \in \mathbb{C} \setminus \Gamma$ .
- $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & e^{-nV(z)} \\ 0 & 1 \end{pmatrix}$ , for  $z \in \Gamma$ ,
- $Y(z) = (I + \mathcal{O}(\frac{1}{z})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ , as  $z \rightarrow \infty$ .

As before,  $\Gamma$  is the contour  $\Gamma = \gamma_1 \cup \gamma \cup \gamma_2$  consisting of the critical trajectory  $\gamma$  and its analytic extensions  $\gamma_1$  and  $\gamma_2$ . See Figure 3.

This Riemann–Hilbert problem has a unique solution if and only if the monic polynomial  $P_n(z)$ , orthogonal with respect to the weight function  $w(z)$ , exists uniquely. If additionally  $P_{n-1}(z)$  exists, then the solution of the Riemann–Hilbert problem is given by:

$$Y(z) = \begin{pmatrix} P_n(z) & (\mathcal{C}P_n w)(z) \\ -2\pi i \gamma_{n-1} P_{n-1}(z) & -\gamma_{n-1} (\mathcal{C}P_{n-1} w)(z) \end{pmatrix},$$

where

$$(\mathcal{C}f)(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)}{s-z} ds \quad (5.1)$$

is the Cauchy transform on  $\Gamma$ , and the coefficient  $\gamma_{n-1}$  is defined as

$$\gamma_{n-1} = \left[ \int_{\Gamma} P_{n-1}^2(s) w(s) ds \right]^{-1}.$$

## 5.2 First transformation

The first transformation  $Y \mapsto T$  is a normalization at  $\infty$ . We use the functions  $\phi_1$  and  $\phi_2$  as in (2.5) and (2.6) which are analytic in  $\mathbb{C} \setminus (\gamma_1 \cup \gamma)$  and  $\mathbb{C} \setminus (\gamma_2 \cup \gamma)$  respectively and satisfy  $\phi_2(z) - \phi_1(z) = \pm \pi i$  for  $z \in \mathbb{C} \setminus \Gamma$ . We set

$$T(z) = \begin{pmatrix} e^{nl} & 0 \\ 0 & e^{-nl} \end{pmatrix} Y(z) \begin{pmatrix} e^{n[\phi_2(z) - \frac{1}{2}V(z)]} & 0 \\ 0 & e^{-n[\phi_2(z) - \frac{1}{2}V(z)]} \end{pmatrix}. \quad (5.2)$$

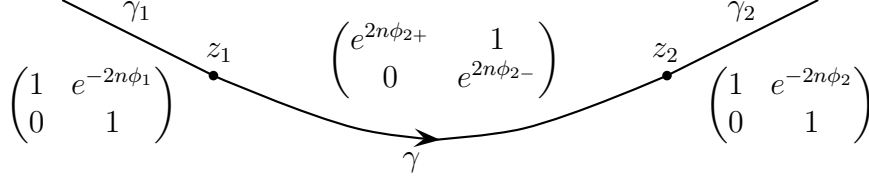


Figure 6: The jump matrices for the Riemann-Hilbert problem for  $T$ .

Now, using (4.6), we obtain by direct integration from (2.6) that

$$\phi_2(z) = \frac{1}{2}V(z) - \log(z) - l + \mathcal{O}\left(\frac{1}{z}\right), \quad z \rightarrow \infty, \quad (5.3)$$

for some constant of integration  $l$ . It follows that

$$e^{n[\phi_2(z) - \frac{1}{2}V(z)]} = z^{-n}e^{-nl} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right), \quad z \rightarrow \infty.$$

Hence  $T$  satisfies the following Riemann–Hilbert problem:

- $T(z)$  is analytic for  $z$  in  $\mathbb{C} \setminus \Gamma$ ;
- $T$  has the jumps indicated in Figure 6;
- $T(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$  as  $z \rightarrow \infty$ .

### 5.3 Second transformation

The second transformation of the Riemann-Hilbert problem is the so-called opening of lenses. From the Cauchy–Riemann equations, it is possible to show that the sign pattern for  $\operatorname{Re} \phi_2$  is as shown in Figure 7. Since  $\phi_1 = \phi_2 \pm \pi i$ , the sign pattern for  $\operatorname{Re} \phi_1$  is exactly the same.

In the second transformation we open a lens-shaped region around  $\gamma$  as in Fig. 8, so that the lens is contained in the region where  $\operatorname{Re} \phi_2 < 0$ :

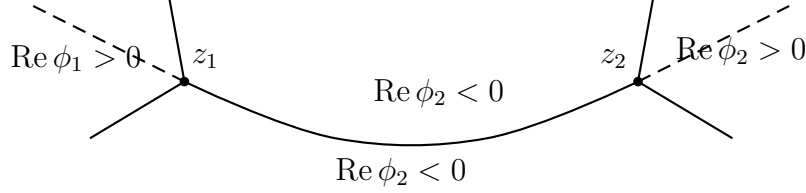


Figure 7: The sign of  $\operatorname{Re} \phi_1 = \operatorname{Re} \phi_2$  in various parts of the complex plane. The solid curves are where  $\operatorname{Re} \phi_2 = 0$ . The curves  $\gamma_1$  and  $\gamma_2$  are shown with dashed lines. We have that  $\phi_1$  is real and positive on  $\gamma_1$  and  $\phi_2$  is real and positive on  $\gamma_2$ .

We define

$$S = \begin{cases} T \begin{pmatrix} 1 & 0 \\ -e^{2n\phi_2} & 1 \end{pmatrix} & \text{in the upper part of the lens,} \\ T \begin{pmatrix} 1 & 0 \\ e^{2n\phi_2} & 1 \end{pmatrix} & \text{in the lower part of the lens,} \\ T & \text{elsewhere.} \end{cases} \quad (5.4)$$

Then  $S$  satisfies the following Riemann–Hilbert problem:

- $S(z)$  is analytic for  $z \in \mathbb{C} \setminus \Gamma_S$ , where  $\Gamma_S$  consists of  $\Gamma$  plus the lips of the lens;
- $S$  has the jumps indicated in Figure 8;
- $S(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$  as  $z \rightarrow \infty$ .

## 5.4 Construction of parametrices

### 5.4.1 Global parametrix

Now we seek an approximation to  $S$  that is valid for large  $n$ . The approximation will consist of two parts, a global parametrix  $N$  away from the endpoints  $z_1$  and  $z_2$  and local parametrices  $P$  at  $z_1$  and  $z_2$ .

The global parametrix  $N$  satisfies a Riemann–Hilbert problem with the same constant jump on  $\gamma$ . Then  $R = SN^{-1}$  will be analytic across  $\gamma$ . We

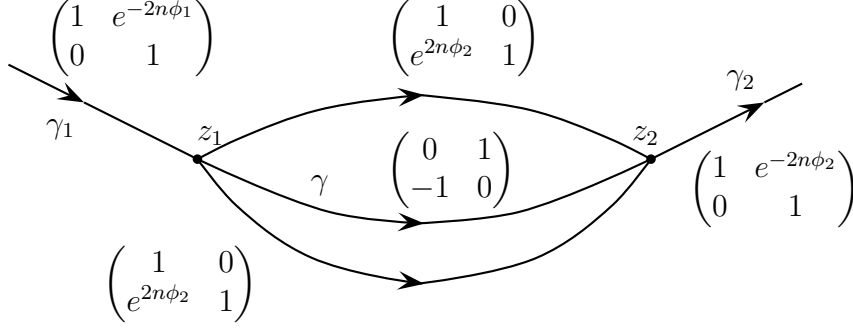


Figure 8: The contour  $\Gamma_S$  and the jump matrices on  $\Gamma_S$  in the Riemann-Hilbert problem for  $S$ .

define  $N$  as

$$N(z) = \begin{pmatrix} \frac{1}{2}(\beta(z) + \beta(z)^{-1}) & \frac{1}{2i}(\beta(z) - \beta(z)^{-1}) \\ -\frac{1}{2i}(\beta(z) - \beta(z)^{-1}) & \frac{1}{2}(\beta(z) + \beta(z)^{-1}) \end{pmatrix}, \quad (5.5)$$

where  $\beta$  is given by (2.10), see [9], [10] and [16].

Then  $N$  satisfies the following Riemann-Hilbert problem:

- $N(z)$  is analytic for  $z \in \mathbb{C} \setminus \gamma$ ;
- $N_+ = N_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  on  $\gamma$ ;
- $N(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$  as  $z \rightarrow \infty$ ;
- $N(z) = \mathcal{O}(|z - z_j|^{-1/4})$  as  $z \rightarrow z_j$  for  $j = 1, 2$ .

It is clear that  $N$  cannot be a good approximation to  $S$  near the endpoints of  $\gamma$ , since it blows up at  $z = z_1$  and  $z = z_2$ , while  $S$  remains bounded there. For this reason we need a different local approximation near the endpoints.

## 5.5 Local parametrix

The local parametrix  $P$  is constructed in neighbourhoods of the endpoint  $z = z_j$ ,  $j = 1, 2$ , say

$$U_\delta(z_j) = \{z \in \mathbb{C} \mid |z - z_j| < \delta\}, \quad j = 1, 2,$$

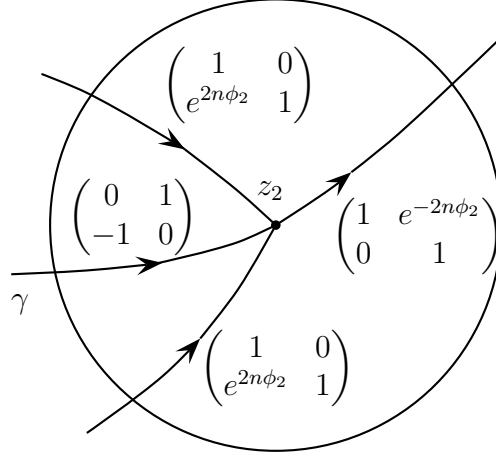


Figure 9: The jump matrices in the Riemann-Hilbert problem for  $P$  defined in the neighbourhood  $U_\delta(z_2)$  of  $z_2$ .

with some small but fixed  $\delta > 0$ . We describe here the construction of  $P$  in  $U_\delta(z_2)$ , the construction in  $U_\delta(z_1)$  being similar.

The local parametrix  $P$  should satisfy the following Riemann-Hilbert problem:

- $\frac{P(z)}{U_\delta(z_2)}$  is analytic for  $z \in U_\delta(z_2) \setminus \Gamma_S$  with a continuous extension to  $U_\delta(z_2) \setminus \Gamma_S$ ;
- $P$  has the jumps on  $\Gamma_S \cap U_\delta(z_2)$  as shown in Fig. 9 (these are the same jump matrices as in the RH problem for  $S$ );
- $P(z) = (I + \mathcal{O}(\frac{1}{n})) N(z)$  as  $n \rightarrow \infty$ , uniformly for  $z \in \partial U_\delta(z_2)$ ;
- $P(z)$  remains bounded as  $z \rightarrow z_2$ .

The construction of  $P$  is given in terms of the Airy function  $\text{Ai}$  and its derivative. We put

$$P(z) = E_n(z) A(n^{2/3} f(z)) \begin{pmatrix} e^{n\phi_2(z)} & 0 \\ 0 & e^{-n\phi_2(z)} \end{pmatrix}, \quad (5.6)$$

where  $A(\zeta)$ ,  $f(z)$ , and  $E_n(z)$  are described below.

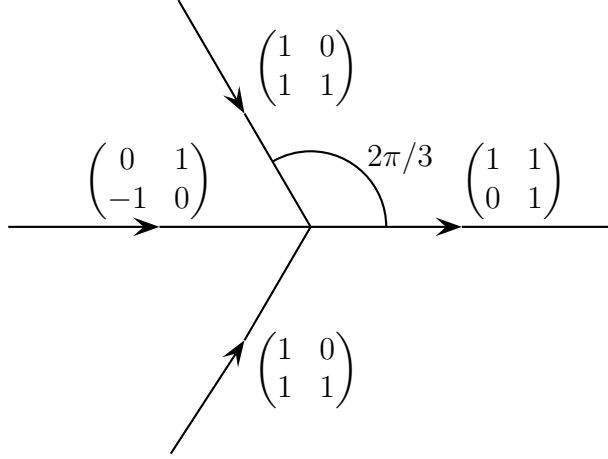


Figure 10: Contours and jump matrices in the Riemann-Hilbert problem for Airy functions.

**Airy parametrix**  $A(\zeta)$  The matrix-valued function  $A(\zeta)$  is the solution of the Airy Riemann-Hilbert problem, which is posed on four infinite rays in an auxiliary  $\zeta$ -plane as follows:

- $A(\zeta)$  is analytic for  $\zeta \in \mathbb{C}$ ,  $\arg \zeta \notin \{0, 2\pi/3, -2\pi/3, \pi\}$ ;
- $A$  has the jumps on the four rays as shown in Fig. 10;
- As  $\zeta \rightarrow \infty$ , we have

$$A(\zeta) = \begin{pmatrix} \zeta^{-1/4} & 0 \\ 0 & \zeta^{1/4} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left( I + \mathcal{O}\left(\frac{1}{\zeta^{3/2}}\right) \right) \begin{pmatrix} e^{-\frac{2}{3}\zeta^{3/2}} & 0 \\ 0 & e^{\frac{2}{3}\zeta^{3/2}} \end{pmatrix} \quad (5.7)$$

- $A(\zeta)$  remains bounded as  $\zeta \rightarrow 0$ .

The solution of this Riemann-Hilbert problem is given by the Airy function  $\text{Ai}(\zeta)$  and rotated versions of it, see [10] and [1, Sec. 10.4]. Let

$$y_0(\zeta) = \text{Ai}(\zeta), \quad y_1(\zeta) = \omega \text{Ai}(\omega\zeta), \quad y_2(\zeta) = \omega^2 \text{Ai}(\omega^2\zeta),$$



where  $\omega = e^{\frac{2\pi i}{3}}$ . These are three solutions of the Airy differential equation  $y'' = \zeta y$  satisfying the connection formula  $y_0(\zeta) + y_1(\zeta) + y_2(\zeta) = 0$ . For instance, in the sector  $0 < \arg \zeta < 2\pi/3$  we have

$$A(\zeta) = \sqrt{2\pi} \begin{pmatrix} y_0(\zeta) & -y_2(\zeta) \\ -iy'_0(\zeta) & iy'_2(\zeta) \end{pmatrix}. \quad (5.8)$$

The solution in the other sectors is obtained from this by applying the appropriate jump matrices.

**Conformal map  $f(z)$**  The map  $f(z)$  is defined by

$$f(z) = \left[ \frac{3}{2} \phi_2(z) \right]^{2/3}, \quad (5.9)$$

which is a conformal map in a neighbourhood of  $z = z_2$ . It is assumed that  $\delta > 0$  is sufficiently small so that  $f$  is indeed a conformal map on  $U_\delta(z_2)$ , and also that the lens around  $\gamma$  is opened in such a way that the lips of the lens inside  $U_\delta(z_2)$  are mapped by  $\zeta = f(z)$  to the rays  $\arg \zeta = \pm 2\pi/3$ . This can be done without any loss of generality.

**Analytic prefactor  $E_n(z)$**  The prefactor  $E_n(z)$  in (5.6) is defined by

$$\begin{aligned} E_n(z) &= N(z) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} n^{1/6} f(z)^{1/4} & 0 \\ 0 & n^{-1/6} f(z)^{-1/4} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} n^{1/6} f(z)^{1/4} \beta^{-1}(z) & 0 \\ 0 & n^{-1/6} f(z)^{-1/4} \beta(z) \end{pmatrix}, \end{aligned} \quad (5.10)$$

which is analytic in  $U_\delta(z_2)$ . It is chosen so that the matching condition  $P(z) = (I + \mathcal{O}(1/n))N(z)$  for  $z \in \partial U_\delta(z_2)$  is satisfied.

Then with these definitions it can be shown that  $P$  defined by (5.6) indeed satisfies the Riemann-Hilbert problem for  $P$ .

## 5.6 Third transformation

In the third and final transformation we use the global parametrix  $N(z)$  and the local parametrices  $P(z)$  to define

$$R(z) = \begin{cases} S(z)N(z)^{-1}, & z \in \mathbb{C} \setminus (\Gamma_S \cup \overline{U_\delta(z_1)} \cup \overline{U_\delta(z_2)}), \\ S(z)P(z)^{-1}, & z \in (\overline{U_\delta(z_1)} \cup \overline{U_\delta(z_2)}) \setminus \Gamma_S. \end{cases} \quad (5.11)$$

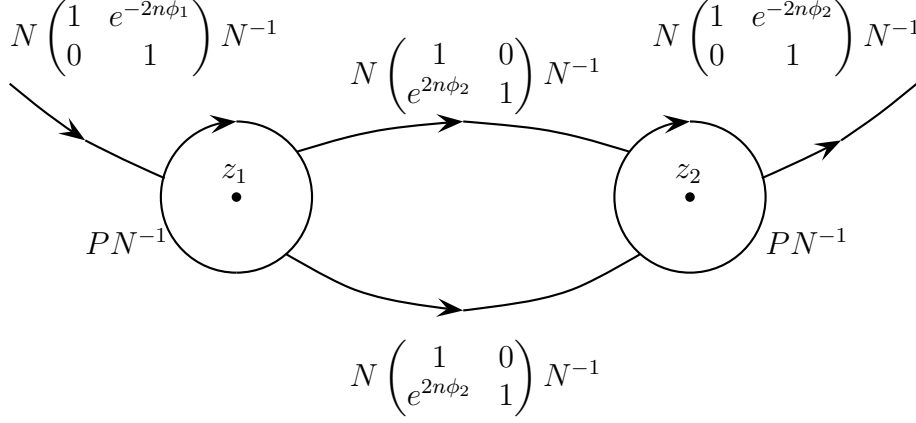


Figure 11: The contour  $\Gamma_R$  and the jump matrices on  $\Gamma_R$  in the Riemann-Hilbert problem for  $R$ .

Then  $R$  has an analytic continuation across  $\gamma$  and across the parts of  $\Sigma_S$  that are inside the disks  $U_\delta(z_1)$  and  $U_\delta(z_2)$ . It satisfies the following Riemann-Hilbert problem:

- $R(z)$  is analytic for  $z \in \mathbb{C} \setminus \Gamma_R$ , where  $\Gamma_R$  is the contour shown in Fig. 11;
- $R$  has jumps on each part of  $\Gamma_R$  with jump matrices as indicated in Fig. 11;
- $R(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$  as  $z \rightarrow \infty$ .

The jump matrices in the Riemann-Hilbert problem for  $R$  tend to the identity matrix as  $n \rightarrow \infty$ . Indeed, since  $P(z) = (I + \mathcal{O}(1/n))N(z)$  as  $n \rightarrow \infty$ , uniformly for  $z \in \partial U_\delta(z_1) \cup \partial U_\delta(z_2)$ , we have that

$$R_+(z) = R_-(z) \left( I + \mathcal{O}\left(\frac{1}{n}\right) \right), \quad z \in \partial U_\delta(z_1) \cup \partial U_\delta(z_2)$$

as  $n \rightarrow \infty$ . On the remaining parts of  $\Gamma_R$  we even have for some positive constant  $c > 0$ ,

$$R_+(z) = R_-(z) \left( I + \mathcal{O}(e^{-cn}) \right), \quad z \in \Gamma_R \setminus (\partial U_\delta(z_1) \cup \partial U_\delta(z_2)),$$

as  $n \rightarrow \infty$ . Thus the jumps on  $R$  tend to the identity matrix uniformly, and in fact also in  $L^2(\Gamma_R)$ .

It then follows from the general theory, see [9] and [16], that the solution to the Riemann-Hilbert problem for  $R$  exists for all large enough  $n$  with

$$R(z) = I + \mathcal{O}\left(\frac{1}{n}\right), \quad \text{as } n \rightarrow \infty, \quad (5.12)$$

uniformly for  $z \in \mathbb{C} \setminus \Gamma_R$ .

## 5.7 Proof of Theorem 2.4

Once we arrive at this result for  $R$ , it is possible to reverse all the transformations  $Y \mapsto T \mapsto S \mapsto R$ , since they are all explicit and invertible. The first thing that follows is that the original Riemann-Hilbert problem for  $Y$  has a unique solution for large enough  $n$ . Since

$$P_n(z) = Y_{11}(z),$$

this proves that the orthogonal polynomials  $P_n$  indeed exist for every large enough  $n$ .

The asymptotic formula (5.12) for  $R$  further yields the first term in an asymptotic expansion of  $Y$  as  $n \rightarrow \infty$ . Following the effect of the inverse transformations  $R \mapsto S \mapsto T \mapsto Y$  on the asymptotic formula (5.12) for  $R$ , we obtain the asymptotics of  $Y$  and therefore of  $P_n$  in the various regions of the complex plane. This will give the different parts of Theorem 2.4.

### 5.7.1 Proof of part (a)

Let  $z \in \mathbb{C} \setminus \gamma$ . We then may and do assume that the lens around  $\gamma$  and the neighborhoods  $U_\delta(z_1)$  and  $U_\delta(z_2)$  are chosen so that  $z$  is in the outside region.

From (5.11) we have that  $S(z) = R(z)N(z)$ . Also, because we are outside the lens, we have  $S(z) = T(z)$  from (5.4) and  $T(z)$  in terms of  $Y(z)$  follows from (5.2). Combining all this we find

$$Y(z) = \begin{pmatrix} e^{-nl} & 0 \\ 0 & e^{nl} \end{pmatrix} \left( I + \mathcal{O}\left(\frac{1}{n}\right) \right) N(z) \begin{pmatrix} e^{-n[\phi_2(z) - \frac{1}{2}V(z)]} & 0 \\ 0 & e^{n[\phi_2(z) - \frac{1}{2}V(z)]} \end{pmatrix}.$$

Then, for the  $(1, 1)$ -entry the first part of the theorem follows in a straightforward way, since

$$g(z) = \frac{1}{2}V(z) - \phi_2(z) - l.$$

### 5.7.2 Proof of part (b)

For  $z$  inside the lens, but outside of the two disks, we have  $S(z) = R(z)N(z)$  as before. From (5.4) we then get

$$T(z) = S(z) \begin{pmatrix} 1 & 0 \\ \pm e^{2n\phi_2} & 1 \end{pmatrix} = R(z)N(z) \begin{pmatrix} 1 & 0 \\ \pm e^{2n\phi_2} & 1 \end{pmatrix}$$

where the  $+$  sign ( $-$  sign) is taken in the upper (lower) part of the lens. Using (5.2) and (5.12) we then find

$$Y(z) = \begin{pmatrix} e^{-nl} & 0 \\ 0 & e^{nl} \end{pmatrix} \left( I + \mathcal{O}\left(\frac{1}{n}\right) \right) N(z) \begin{pmatrix} e^{-n[\phi_2(z) - \frac{1}{2}V(z)]} & 0 \\ \pm e^{n[\phi_2(z) + \frac{1}{2}V(z)]} & e^{n[\phi_2(z) - \frac{1}{2}V(z)]} \end{pmatrix}.$$

Then for the  $(1, 1)$ -entry we obtain from this

$$\begin{aligned} P_n(z) &= Y_{11}(z) = e^{n[\frac{V(z)}{2} - l]} \left( 1 + \mathcal{O}\left(\frac{1}{n}\right), \mathcal{O}\left(\frac{1}{n}\right) \right) N(z) \begin{pmatrix} e^{-n\phi_2(z)} \\ \pm e^{n\phi_2(z)} \end{pmatrix} \\ &= e^{n[\frac{V(z)}{2} - l]} \left( e^{-n\phi_2(z)} N_{11}(z) \pm e^{n\phi_2(z)} N_{12}(z) + \mathcal{O}\left(\frac{1}{n}\right) \right) \end{aligned}$$

as  $n \rightarrow \infty$ . This proves part (b) of the theorem.

## 5.8 Proof of part (c)

In the neighbourhoods  $U_\delta(z_1)$  and  $U_\delta(z_2)$  of the endpoints  $z_1$  and  $z_2$  we use the local parametrix  $P(z)$  to obtain an approximation for  $P_n(z)$  in terms of Airy functions. Indeed, by (5.11) and (5.12),

$$S(z) = R(z)P(z) = \left( I + \mathcal{O}\left(\frac{1}{n}\right) \right) P(z)$$

for  $z \in U_\delta(z_1) \cup U_\delta(z_2)$ . If we assume that  $z$  is inside the disk  $U_\delta(z_2)$  but outside the lens around  $\gamma$ , then we find by following the transformations (5.4)

and (5.2) that

$$Y(z) = \begin{pmatrix} e^{-nl} & 0 \\ 0 & e^{nl} \end{pmatrix} \left( I + \mathcal{O}\left(\frac{1}{n}\right) \right) P(z) \begin{pmatrix} e^{-n[\phi_2(z) - \frac{V(z)}{2}]} & 0 \\ 0 & e^{n[\phi_2(z) - \frac{V(z)}{2}]} \end{pmatrix}.$$

Using (5.6) and (5.10), we obtain from this that

$$\begin{aligned} Y(z) &= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-nl} & 0 \\ 0 & e^{nl} \end{pmatrix} \left( I + \mathcal{O}\left(\frac{1}{n}\right) \right) \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} n^{1/6} f(z)^{1/4} \beta(z)^{-1} & 0 \\ 0 & n^{-1/6} f(z)^{-1/4} \beta(z) \end{pmatrix} \\ &\quad \times A(n^{2/3} f(z)) \begin{pmatrix} e^{n \frac{V(z)}{2}} & 0 \\ 0 & e^{-n \frac{V(z)}{2}} \end{pmatrix}. \end{aligned}$$

To evaluate  $A(n^{2/3} f(z))$  we use (5.8) and it follows that

$$\begin{aligned} P_n(z) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \sqrt{\pi} e^{n[\frac{V(z)}{2} - l]} \left( 1 + \mathcal{O}\left(\frac{1}{n}\right), \mathcal{O}\left(\frac{1}{n}\right) \right) \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} n^{1/6} f(z)^{1/4} \beta(z)^{-1} & 0 \\ 0 & n^{-1/6} f(z)^{-1/4} \beta(z) \end{pmatrix} \begin{pmatrix} \text{Ai}(n^{2/3} f(z)) \\ -i \text{Ai}'(n^{2/3} f(z)) \end{pmatrix} \end{aligned}$$

as  $n \rightarrow \infty$ . This proves part (c) of the theorem in case  $z \in U_\delta(z_2)$  is outside the lens. A similar calculation leads to the same expression in case  $z$  is inside the lens. This completes the proof of part (c) of Theorem 2.4.

## 6 Concluding remarks

We have presented a Riemann–Hilbert analysis of a family of polynomials orthogonal with respect to a varying exponential weight on certain curves of the complex plane. The problem was motivated by the fact that the zeros of these polynomials are complex Gaussian quadrature points for an oscillatory integral on an interval  $[a, b] \subset \mathbb{R}$ . The zeros cluster on analytic arcs in the complex plane, which are given by a critical trajectory of a suitable quadratic differential.

We have focused on the case where the weight function is  $V(z) = -iz^3/3$ , for which we were able to obtain explicit expressions throughout the Riemann–Hilbert analysis. A similar procedure (with more complicated computations) can be applied in principle to the more general case  $V(z) = -iz^r/r$  with  $r \geq 5$  and odd. The only difficulty is the determination of a curve with the  $S$ -property in this more general case. It would be interesting to know if we are in the one-cut case for every odd  $r$ .

It is also worth remarking that the Riemann–Hilbert analysis can provide more detailed asymptotic information than the one given before, following the ideas exposed in [10], [20]. The importance of these results from a numerical point of view is currently under investigation.

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